All-around: Probability and statistics

Problem 1. Suppose that a study has N subjects, divided into two groups. For subject $i (i = 1, ..., N)$, let Z_i be the group assignment, with $Z_i = 1$ being the treatment group and $Z_i = 0$ being the control group, and X_i be the covariates. The propensity score of subject i is the probability of that subject being in the treatment group: $e(X_i) = P(Z_i = 1 \mid X_i)$. A metric for the similarity of the covariates distribution between the groups is the Bhattacharyya coefficient:

$$\phi \equiv \int_0^1 \sqrt{f_1(u) f_0(u)} \mathrm{d}u,$$

where $f_z(u)$ is the density of the propensity score in group z (for $z = 0, 1$). Assume $e(X) \sim \mathsf{Beta}(a, b)$, derive the Bhattacharyya coefficient ϕ as a function of (a, b). (Note: The pdf of Beta distribution: $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$, where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function).

Problem 2. Let $X_t := e^{B_t - \frac{t}{2}}$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion with $B_0 = 0$. Find the distribution of $M := \sup_{t \geq 0} X_t$.